Chapter 1

What Is Calculus, Anyway?

In This Chapter

- Why calculus is useful
- The historic origins of calculus
- The authorship controversy
- Can I ever learn this?

The word *calculus* can mean one of two things: a computational method or a mineral growth in a hollow organ of the body, such as a kidney stone. Either definition often personifies the pain and anguish endured by students trying to understand the subject. It is far from controversial to suggest that mathematics is not the most popular of subjects in contemporary education, but calculus holds the great distinction of King of the Evil Math Realm, especially by the math phobic. It represents an unattainable goal, an unthinkable miasma of confusion and complication, and few venture into its realm unless propelled by such forces as job advancement or degree requirement. No one knows how much people fear calculus more than a calculus teacher.

The minute people find out that I taught a calculus class, they are compelled to describe, in great detail, exactly how they did in high school math, what subject they "topped out" in, and why they feel that calculus is the embodiment of evil. Most of these people are my barbers, and I can't explain why. All of the friendly folks at the Hair Cuttery have come to know me as the strange balding man with arcane and baffling mathematical knowledge.

Most of the fears surrounding calculus are unjustified. Calculus is a step up from high school algebra, no more. Following a straightforward list of steps, just like you do with most algebra problems, solves the majority of calculus problems. Don't get me wrong—calculus is not always easy, and the problems are not always trivial, but it is not as imposing as it seems. Calculus is a truly fascinating tool with innumerable applications to "real life," and for those of you who like soap operas, it's got one of the biggest controversies in history to its credit.

What's the Purpose of Calculus?

Calculus is a very versatile and useful tool, not a one-trick pony by any stretch of the imagination. Many of its applications are direct upgrades from the world of algebra—methods of accomplishing similar goals, but in a far greater number of situations. Whereas it would be impossible to list all the uses of calculus, the following list represents some interesting highlights of the things you will learn by the end of the book.

Critical Point

What we call "calculus," scholars call "*the* calculus." Because any method of computation can be called a calculus and the discoveries comprising modern-day calculus are so important, the distinction is made to clarify. I personally find the terminology a little pretentious and won't use it. I've never been asked "Which calculus are you talking about?"

Finding the Slopes of Curves

One of the earliest algebra topics learned is how to find the slope of a line—a numerical value that describes just how slanted that line is. Calculus affords us a much more generalized method of finding slopes. With it, we can find not only how steeply a line slopes, but indeed, how steeply any curve slopes at any given time. This might not at first seem useful, but it is actually one of the most handy mathematics applications around.

Calculating the Area of Bizarre Shapes

Without calculus, it is difficult to find areas of shapes other than those whose formulas you learned in geometry. Sure, you may be a pro at finding the area of a circle, square, rectangle, or triangle, but how would you find the area of a shape like the one shown in Figure 1.1?

Justifying Old Formulas

There was a time in your math career that you took formulas on faith. Sometimes we still need to do that, but calculus affords us the opportunity to finally verify some of those old formulas, especially from geometry. You were always told that the volume of a cone was one third the volume of a cylinder with the same Vadius $\left(V = \frac{1}{3}\pi r^2 h\right)$, but through a simple calculus process of three-dimensional linear rotation, we can finally prove it. (By the way, the process is simple even though it may not sound like it right now.)

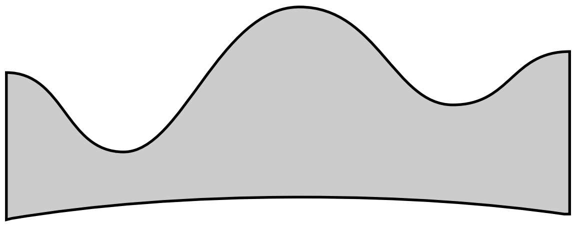


Figure 1.1

Calculate this area? We're certainly not in Kansas anymore ….

Calculate Complicated *X*-Intercepts

Without the aid of a graphing calculator, it is exceptionally hard to calculate an *irrational root*. However, a simple, repetitive process called Newton's Method (named after Sir Isaac Newton) allows you to calculate an irrational root to whatever degree of accuracy you desire.

Talk the Talk

An **irrational root** is an *x*-intercept that is not a fraction. Fractional (rational) roots are much easier to find, because you can typically factor the expression to calculate them, a process that is taught in the earliest algebra classes. No good, generic process of finding irrational roots is possible until you use calculus.

Visualizing Graphs

You may already have a good grasp of lines and how to visualize their graphs easily, but what about the graph of something like $y = x^3 + 2x^2 - x + 1$? Very elementary calculus tells you exactly where that graph will be increasing, decreasing, and twisting. In fact, you can find the highest and lowest points on the graph without plotting a single point.

Finding the Average Value of a Function

Anyone can average a set of numbers, given the time and the fervent desire to divide. Calculus allows you to take your averaging skills to an entirely new level. Now you can even find, on average, what height a function travels over a period of time. For example, if you graph the path of an airplane (see Figure 1.2), you can calculate its average cruising altitude with little or no effort. Determining its average velocity and acceleration are no harder. You may never have had the impetus to do such a thing, but you've got to admit that it's certainly more interesting than averaging the odd numbers less than 50.

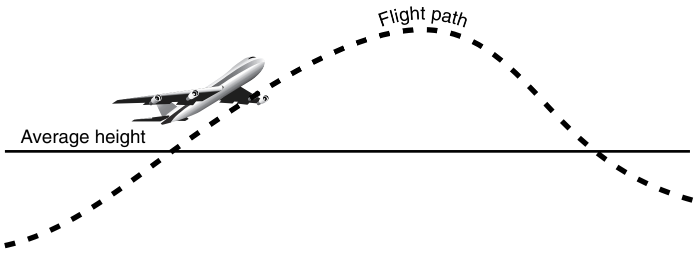


Figure 1.2

Even though this plane's flight path is not defined by a simple shape (like a semicircle), using calculus you can calculate all sorts of things, like its average altitude during the journey or the number of complementary peanuts you dropped when you fell asleep.

Calculating Optimal Values

One of the most mind-bendingly useful applications of calculus is the optimization of functions. In just a few steps, you can answer questions such as: "If I have 1,000 feet of fence, what is the largest rectangular yard I can make?" or "Given a rectangular sheet of paper which measures 8.5 inches by 11 inches, what are the dimensions of the box I can make containing the greatest volume?" The traditional way to create an open box from a rectangular surface is to cut congruent squares from the corners of the rectangle and then to fold the resulting sides up, as shown in Figure 1.3.

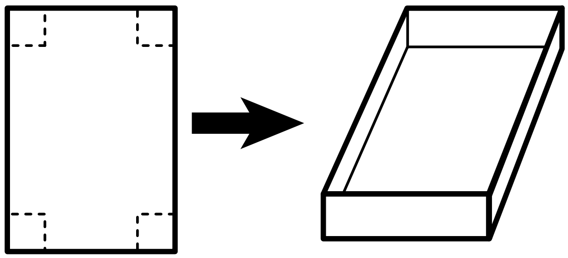


Figure 1.3

With a few folds and cuts, you can easily create an open box from a rectangular surface.

I tend to think of learning calculus and all of its applications as suddenly growing a third arm. Sure, it may feel funny having a third arm at first. In fact, it'll probably make you stand out in bizarre ways from those around you. However, given time, you're sure to find many uses for that arm that you'd have never imagined without having first possessed it.

Who's Responsible for This?

Tracking the discovery of calculus is not as easy as, say, tracking the discovery of the safety pin. Any new mathematical concept is usually the result of hundreds of years of investigation, debate, and debacle. Many come close to stumbling upon key concepts, but only the lucky few who finally make the small, key connections receive the credit. Such is the case with calculus.

Calculus is usually defined as the combination of the differential and integral techniques you will learn later in the book. However, historical mathematicians would never have swallowed the concepts we take for granted today. The key ingredient missing in mathematical antiquity was the hairy notion of infinity. Mathematicians and philosophers of the time had an extremely hard time conceptualizing infinitely small or large quantities. Take, for instance, the Greek philosopher Zeno.

Ancient Influences

Zeno took a very controversial position in mathematical philosophy: He argued that all motion is impossible. In the paradox titled Dichotomy, he used a compelling, if not strange, argument illustrated in Figure 1.4.

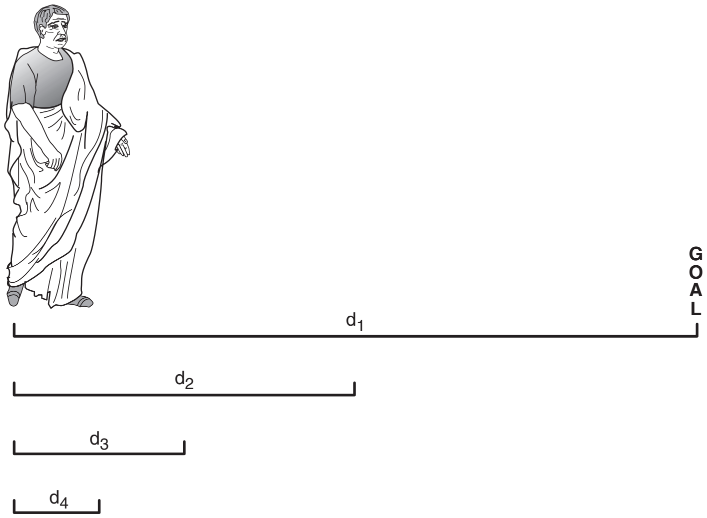


Figure 1.4

The infinite subdivisions described in Zeno's Dichotomy.

Critical Point

The most famous of Zeno's paradoxes is a race between a tortoise and the legendary Achilles called, appropriately, *the Achilles*. Zeno contends that if the tortoise has a head start, no matter how small, Achilles will never be able to close the distance. To do so, he'd have to travel half of the distance separating them, then half of that, ad nauseum, presenting the same dilemma illustrated by the Dichotomy.

In Zeno's argument, the individual pictured wants to travel to the right, to his eventual destination. However, before he can travel that distance (d_1), he must first travel half of that distance (d_2). That makes sense, since d_2 is smaller and comes first in the path. However, before the d_2 distance can be completed, he must first travel half of *it* (d_3). This procedure can be repeated indefinitely, which means that our beleaguered sojourner must travel an infinite number of distances. No one can possibly do an infinite number of things in a finite amount of time, says Zeno, since an infinite list will never be exhausted. Therefore, not only will the man never reach his destination, he will, in fact, never start moving at all! This could account for the fact that you never seem to get anything done on Friday afternoons.

Critical Point

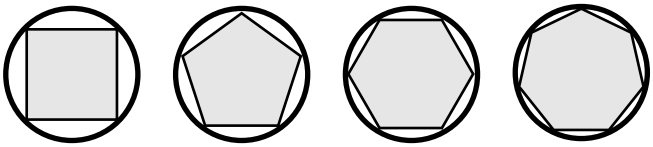
In case the suspense is killing you, let me ruin the ending for you. The essential link to completing calculus and satisfying everyone's concerns about infinite behavior was the concept of limit, which laid the foundation for both derivatives and integrals.

Zeno didn't actually believe that motion was impossible. He just enjoyed challenging the theories of his contemporaries. What he, and the Greeks of his time, lacked was a good understanding of infinite behavior. It was unfathomable that an innumerable number of things could fit into a measured, fixed space. Today, geometry students accept that a line segment, though possessing fixed length, contains an infinite number of points. The development of some reasonable and yet mathematically sound concept of very large quantities or very small quantities was required before calculus could sprout.

Some ancient mathematicians weren't troubled by the apparent contradiction of an infinite amount in a finite space. Most notably, Euclid and Archimedes contrived the method of exhaustion as a technique to finding the area of a circle, since the exact value of π wouldn't be around for some time. In this technique, regular polygons were inscribed in a circle; the higher the number of sides of the polygon, the closer the area of the polygon would be to the area of the circle (see Figure 1.5).

Figure 1.5

The higher the number of sides, the closer the area of the inscribed polygon approximates the area of the circle.



In order for the method of exhaustion (which is aptly titled, in my opinion) to give the exact value for the circle, the polygon would have to have an infinite number of sides. Indeed, this magical incarnation of geometry can only be considered theoretically, and the idea that a shape of infinite sides could have a finite area made most people of the time very antsy. However, seasoned calculus students of today can see this as a simple limit problem. As the number of sides approaches infinity, the area of the polygon approaches πr^2, where r is the radius of the circle. Limits are essential to the development of both the derivative and integral, the two fundamental components of calculus. Although Newton and Leibniz were unearthing the major discoveries of calculus in the late 1600s and early 1700s, no one had established a formal limit definition. Although this may not keep *us* up at night, it was, at the least, troubling at the time. Mathematicians worldwide started sleeping more soundly at night circa 1751, when Jean Le Rond d'Alembert wrote *Encyclopédie* and established the formal definition of the limit. The delta-epsilon definition of the limit we use today is very close to that of d'Alembert.

Even before its definition was established, however, Newton had given a good enough shot at it that calculus was already taking shape.

Newton vs. Leibniz

Sir Isaac Newton, who was born in poor health in 1642 but became a world-renowned smart guy (even during his own time), once retorted, "If I have seen farther than Descartes, it is because I have stood on the shoulders of giants." No truer thing could be said about any major mathematical discovery, but let's not give the guy too much credit for his supposed modesty … more to come on that in a bit. Newton realized that infinite series (e.g., the method of exhaustion) were not only great approximators, but if allowed to actually reach infinity, they gave the exact values of the functions they approximated. Therefore, they behaved according to easily definable laws and restrictions usually only applied to known functions. Most importantly, he was the first person to recognize and utilize the inverse relationship between the slope of a curve and the area beneath it.

That inverse relationship (contemporarily called the Fundamental Theorem of Calculus) marks Newton as the inventor of calculus. He published his findings, and his intuitive definition of a limit, in his 1687 masterwork entitled *Philosophiae Naturalis Principia Mathematica*. The *Principia*, as it is more commonly known today, is considered by some (those who consider such things, I suppose) to be the greatest scientific work of all time, excepting of course any books yet to be written by the comedian Sinbad. Calculus was actively used to solve the major scientific dilemmas of the time:

- Calculating the slope of the tangent line to a curve at any point along its length
- Determining the velocity and acceleration of an object given a function describing its position, and designing such a position function given the object's velocity or acceleration
- Calculating arc lengths and the volume and surface area of solids
- Calculating the relative and absolute *extrema* of objects, especially projectiles

However, with a great discovery often comes great controversy, and such is the case with calculus.

Talk the Talk

Extrema points are high or low points of a curve (maxima or minima, respectively). In other words, they represent extreme values of the graph, whether extremely high or extremely low, in relation to the points surrounding them.

Enter Gottfried Wilhelm Leibniz, child prodigy and mathematical genius. Leibniz was born in 1646 and completed college, earning his Bachelor's degree, at the ripe old age of 17. Because Leibniz was primarily self-taught in the field of mathematics, he often discovered important mathematical concepts on his own, long after someone else had already published them. Newton actually credited Leibniz in his *Principia* for developing a method similar to his. That similar method evolved into a near match of Newton's work in calculus, and in fact, Leibniz published his breakthrough work inventing calculus *before* Newton, although Newton had already made the exact discovery years before Leibniz. Some argue that Newton possessed extreme sensitivity to criticism and was, therefore, slow to publish. The mathematical war was on: Who invented calculus first and thus deserved the credit for solving a riddle thousands of years old?

Critical Point

Ten years after Leibniz's death, Newton erased the reference to Leibniz from the third edition of the *Principia* as a final insult. This is approximately the academic equivalent of Newton throwing a chair at Leibniz on *The Jerry Springer Show* (topic: "You published your solution to an ancient mathematical riddle before me and I'm fightin' mad!").

Today, Newton is credited for inventing calculus first, although Leibniz is credited for its first publication. In addition, the shadow of plagiarism and doubt has been lifted from Leibniz, and it is believed that he discovered calculus completely independent of Newton. However, two distinct factions arose and fought a bitter war of words. British mathematicians sided with Newton, whereas continental Europe supported Leibniz, and the war was long and hard. In fact, British mathematicians were effectively alienated from the rest of the European mathematical community because of the rift, which probably accounts for the fact that there were no great mathematical discoveries made in Britain for some time thereafter.

Although Leibniz just missed out on the discovery of calculus, many of his contributions live on in the language and symbols of mathematics. In algebra, he was the first to use a dot to indicate multiplication ($3 \cdot 4 = 12$) and a colon to designate a proportion ($1:2 = 3:6$). In geometry, he contributed the symbols for congruent ($\cong$) and similar ($\sim$). Most famous of all, however, are the symbols for the derivative and the integral, which we also use.

Will I Ever Learn This?

History aside, calculus is an overwhelming topic to approach from a student's perspective. There are an incredible number of topics, some of which are related, but most of which are not in any obvious sense. However, there is no topic in calculus that is, in and of itself, very difficult once you understand what is expected of you. The real trick is to quickly recognize what sort of problem is being presented and then to attack it using the methods you will read and learn in this book.

I have taught calculus for a number of years, to high school students and adults alike, and I believe that there are four basic steps to being successful in calculus:

Critical Point

Leibniz also coined the term *function*, which is commonly learned in an elementary algebra class. However, most of Leibniz's discoveries and innovations were eclipsed by Newton, who made great strides in the topics of gravity, motion, and optics (among other things). The two men were bitter rivals and were fiercely competitive against each other.

- *Make sure to understand what the major vocabulary words mean.* This book will present all important vocabulary terms in simple English, so you not only understand what the terms mean, but how they apply to the rest of your knowledge.
- *Sift through the complicated wording of the important calculus theorems and strip away the complicated language.* Math is just as foreign a language as French or Spanish to someone who doesn't enjoy numbers, but that doesn't mean you can't understand complicated mathematical theorems. I will translate every theorem into plain English and make all the underlying implications perfectly clear.
- *Develop a mathematical instinct.* As you read, I will help you recognize subtle clues presented by calculus problems. Most problems do everything but tell you exactly how they must be solved. If you read carefully, you will develop an instinct, a feeling that will tingle in your inner fiber and guide you toward the right answers. This comes with practice, practice, practice, so the majority of the book presents sample problems with detailed solutions to help you navigate the muddy waters of calculus.
- *Sometimes you just have to memorize.* There are some very advanced topics covered in calculus that are hard to prove. In fact, many theorems cannot be proven until you take much more advanced math courses. Whenever I think that proving a theorem will help you understand it better, I will do so and discuss it in detail. However, if a formula, rule, or theorem has a proof that I deem unimportant to you mastering the topic in question, I will omit it, and you'll just have to trust me that it's for the best.

The Least You Need to Know

- Calculus is the culmination of algebra, geometry, and trigonometry.
- Calculus as a tool enables us to achieve greater feats than the mathematics courses that precede it.
- Limits are foundational to calculus.
- Newton and Leibniz both discovered calculus independently, though Newton discovered it first.
- With time and dedication, anyone can be a successful calculus student.